The velocity history of any particular system may be obtained from either the analytical solutions in Ref. 1 or numerical integration of Eq. (2). For engineering purposes, a more convenient procedure prevails when the propellant comprises a small part of the total mass, which often is the arrangement. In that event, velocity becomes directly proportional to the impulse expenditure. The necessary knowledge of the impulse history can be derived from Eqs. (1-3) of Ref. 1. It is found from those equations that the time required to deliver some fraction q of the total impulse is $\tau = 1-(1-q)^{(1/\zeta)}$, or, in terms of real time t,

$$\psi t = 1/(1-q)^{(1/\zeta)} - 1 \tag{5}$$

With a given thruster configuration parameter ψ, the variation of impulse may be determined immediately from Eq. (5). Once the impulse distribution has been established, the approximate velocity history becomes $v/v_{\max} = q(t)$, or

$$v/v_{\max} = 1 - [1/(1+\psi t)]^{\zeta} \quad \text{if} \quad \lambda \cong 1 \tag{6}$$

Using Eq. (5), it may be seen that for propulsive gases having, for example, seven active degrees of freedom, and with $\psi = 1/\text{sec}$, 90% of the available impulse is released in about ⅓ sec. An equivalent part of the terminal velocity occurs during that time interval if the mass faction of the working fluid is initially small.

With the preceding information, the ideal characteristics of an extensive group of cold-gas thrusters may be assessed readily.

Acknowledgment

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Optimal Terminal Guidance with Constraints at Final Time

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I. Introduction

RECENT intelligence suggests that the impenetrable nature of heavy armor may be susceptible to missile attacks at a relatively high angle of impact, with respect to the horizon. In many modes of direct encounter, the target may not be reachable with a body pitch attitude angle of the proper magnitude. There are several possible reasons for this condition, including lack of energy (fuel), lack of time to maneuver into the more desirable attitude, or lack of control information by appropriate sensors to command the response. This condition has been recognized for some time at the Missile Research and Development Command, and consequently there have been attempts to modify trajectory shapes by a variety of predetermined control laws. However, there has been a certain lack of robustness in the solutions obtained over the entire range of conditions anticipated. This situation motivated a search for optimal solutions to the guidance problem and a study of tradeoffs among the suboptimal candidates that were deemed feasible.

Terminal guidance schemes for tactical missiles may be based on a classical approach, such as a proportional navigation and guidance law,[1,2] or on a modern control theoretic approach.[3-5] In the latter, a control law is derived in terms of time-varying feedback gains when formulated as a linear quadratic control problem. A suboptimal terminal guidance system for re-entry vehicles, derived using the modern approach, was the basis for the initial work on this problem.

Kim and Grider[6] studied a suboptimal terminal guidance system for a re-entry vehicle by placing a constraint on the body attitude angle at impact. Their problem was oriented to a long-range, high-altitude mission. Their scenario was formulated as a linear quadratic control problem with certain key assumptions. The angle of attack of the re-entry vehicle was assumed to be small and thus was neglected. Furthermore, the autopilot response was assumed to be instantaneous, i.e., with no lag time attributed to the transfer of input commands to output reaction.

These conditions have been studied in an extension of their earlier work.[7] A formulation is given for a system that has finite time delay. In fact, the increase and decrease in time delay has interesting ramifications on the solution. The angle-of-attack assumption is investigated, and, although not solved analytically in closed form, the system is derived.

There is more than just a passing academic interest in this problem. As suggested previously, the antiarmor role of several Army weapon systems very well may be enhanced by this technique. The reduction to a practical implementation or mechanization will be studied and described in a future paper. This paper, however, summarizes the feasibility of the concept.

II. State Representation and Problem Formulation

The geometry of the tactical missile-target position is given in Fig. 1. Assume that the angle of attack is small and thus can be neglected (this assumption will be considered later), and choose the following set of variables:

$$x = \begin{bmatrix} Y_d \\ \dot{Y}_d \\ A_L \\ \theta \end{bmatrix} \equiv \begin{bmatrix} Y_t - Y_m \\ \dot{Y}_t - \dot{Y}_m \\ A_L \\ \theta \end{bmatrix} \tag{1}$$

where Y_d is the position variable from the missile to the target projected on the ground; Y_t is the position variable of the target; Y_m is the position variable of the missile projected on the ground; $\dot{Y}_d$ is the derivative of Y_d, the missile to the target velocity projected on the ground; A_L is the lateral acceleration of the missile; θ is the body attitude angle of the missile; and α is the angle of attack of the missile shown in Fig. 1.

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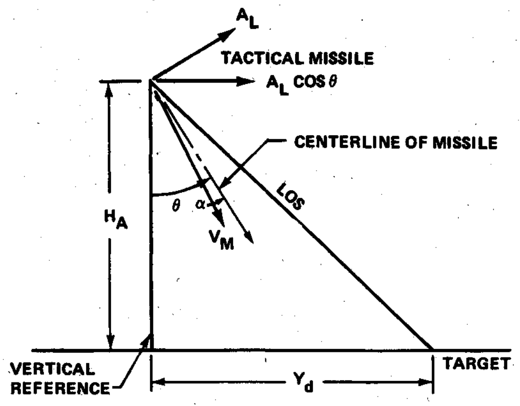

Fig. 1 Geometry of tactical missile target positions.

The system dynamics can be expressed as

$$\dot{Y}_d = \dot{Y}_d \quad \ddot{Y}_d = -A_L \cos\theta \tag{2a}$$

$$\dot{A}_L = -\omega_l A_L + K_l u \quad \dot{\theta} = K_a u \tag{2b}$$

Note that the lag in the autopilot has been represented by a first-order lag network

$$A_L(s)/u(s) = K_l/(s+\omega_l) \tag{3}$$

where u represents the control.

Linearizing about an operating point (i.e., $\cos\theta = b$) and viewing the system in the standard canonical form,

$$\dot{x} = Ax + Bu$$

the result is

$$\begin{bmatrix} \dot{Y}_d \\ \ddot{Y}_d \\ \dot{A}_L \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -b \\ 0 & 0 & 0 & -\omega_l \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Y_d \\ \dot{Y}_d \\ \theta \\ A_L \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ K_l \\ K_a \end{bmatrix} u \tag{4}$$

This optimal control problem will have a controller of the form

$$u = C_Y Y_d + C_{\dot{Y}} \dot{Y}_d + C_\theta \theta + C_{A_L} A_L \tag{5}$$

where C_Y, $C_{\dot{Y}}$, C_θ, C_{A_L} are time-varying coefficients chosen to minimize the cost functional

$$J = Y_d^2(t_f) + \gamma\theta^2(t_f) + \beta \int_{t_0}^{t_f} u^2(t)\,dt \tag{6}$$

Here t_f is the final (impact) time and t_0 is initial time, with γ and β defined as constant weighing factors. The integral term in the performance index of Eq. (6) is used to constrain the total expenditure of u. The actual constraints on miss distance and attitude angle at impact are $|Y_d(t_f)| \leq 5$ ft and $|\theta(t_f)| \leq 5°$.

III. Effectiveness of the Control Law

Using the Euler-Lagrange formulation, a closed-form solution of the controller u was obtained. In an earlier work,[6] a control law of the form

$$\dot{u} = C_Y Y_d + C_{\dot{Y}} \dot{Y}_d + C_\theta \theta$$

was derived under the assumption of zero autopilot lag (i.e., $A_L = K_l/\omega_l u$). The C_Y coefficient obtained in Ref. 6 was

$$C_Y = \frac{[-\beta g(t_f - t) - g\gamma K_a^2 (t_f - t)^2/2]}{\Delta}$$

where $g \equiv -bk_l/\omega_l$ and

$$\Delta \equiv \beta^2 + \gamma\beta K_a^2 (t_f - t) + \beta_g^2 \frac{(t_f - t)^3}{3} + \frac{\gamma g^2 K_a^2 (t_f - t)^4}{12} \tag{7}$$

It can be shown that, as lag in the autopilot tends to zero, i.e., $\omega_l \to \infty$ in the optimal control solution, the control law of Eq. (7) surfaces as the limiting case of the new control law.

Even though acceptable performance is obtained using the control law described by the coefficients in Eq. (7), there is some question regarding the sensitivity of performance due to lag. In particular, this question arises: at what point does performance degenerate to justify the more complex control law to achieve the given performance constraints? This is answered partially by referring to Fig. 2, which is a plot of miss distance vs lag for the two control laws, and Fig. 3, which is attitude angle at impact vs lag.

For lag constraints $\omega_l = 1, 2$, the simulation with the control law of Eq. (7) becomes unstable. This is not true with the referenced optimal control law. For $\omega_l = 3$, a miss distance of 68 ft was obtained, whereas the new control law had a miss distance of 6.1 ft. Acceptable performance for Eq. (7) is not obtained until $\omega_l > 4$. It should be noted that the parameters in that control law were chosen for a nominal lag value of $\omega_l = 5$. Should the amount of lag in the tactical missile autopilot vary and not be approximated closely a priori, then the more complicated control law needs to be implemented. In other words, the control law of Eq. (7) performs adequately only for lag near the assumed nominal value.

The control law implementation was investigated for adaptability to approximation by signals from physically realizable sources. The coefficients C_Y, $C_{\dot{Y}}$ were ap-

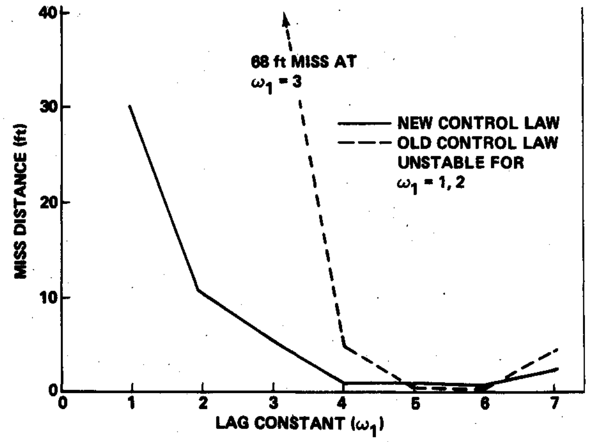

Fig. 2 A comparison of miss distance performance.

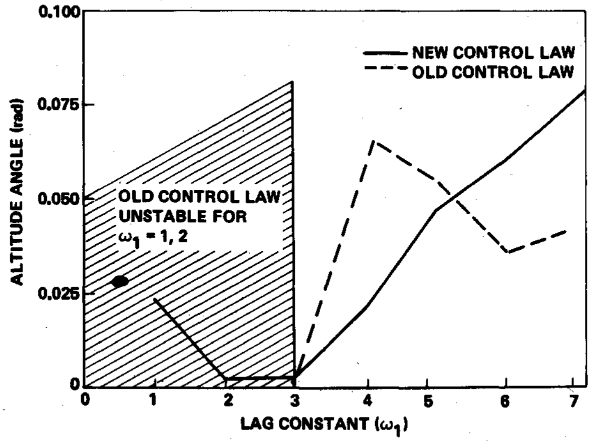

Fig. 3 A comparison of impact angle performance.

proximated by linear functions pieced together at three break points. Performance turned out to be too sensitive to the approximation error, and the performance constraint could not be met. The number of linear segments used was increased, but acceptable performance still not was achieved. It was concluded that successful implementation would require more than merely linear approximations to the time-varying coefficients.

IV. Angle-of-Attack Formulations

The angle of attack probably cannot be ignored for the larger tactical missile. For such a missile, the system of equations should include the angle of attack α. In addition, because it is feasible to achieve only a small angle of attack at impact, a reasonable performance index to be minimized would seem to be

$$J=C_1 Y_d^2(t_f)+C_2\theta^2(t_f)+C_3\alpha^2(t_f)+C_4\int_{t_0}^{t_f} u^2(t)\mathrm{d}t \quad (8)$$

To produce a formulation of the problem which incorporates the angle of attack α, it is assumed that $\dot{\alpha}$ can be expressed as a linear combination of the control u and the attitude rate $\dot{\theta}$. Assuming motion only in the pitch plane, then

$$\dot{\theta}=Q \text{ (pitch rate)}$$

$$\ddot{\theta}=\dot{Q}=(TAB_2+TCB_2)/I_2$$

where TAB_2 is the pitching moment coefficient due to angle of attack and pitch rate, and TCB_2 is the pitching moment coefficient due to fin deflection. Now

$$TAB_2=-qSd(C_{m_\alpha}\alpha+C_{m_\theta}Q)\equiv L_1\alpha+L_2\dot{\theta}$$

where q is the dynamic pressure, S is the missile reference area, and d is the missile reference dimension. Also, noting that pitch fin deflection (δ_q) is equivalent to control (u), $TCB_2=qSdC_{m_\delta}\,u\equiv L_3I_2/u$.

Therefore,

$$\ddot{\theta}=L_1\alpha+L_2\dot{\theta}+L_3u$$

Figure 1 yields the following system for the state variables Y_d, $\dot{Y}_d$, A_L, θ, $\dot{\theta}$, α:

$$\dot{Y}_d=\dot{Y}_d \qquad \ddot{Y}_d=-A_L\cos(\theta-\alpha) \quad (9a)$$

$$\dot{A}_L=-\omega_1A_L+K_1u \qquad \dot{\theta}=\dot{\theta} \quad (9b)$$

$$\ddot{\theta}=L_1\alpha+L_2\dot{\theta}+L_3u \qquad \dot{\alpha}=K_3u+K_4\dot{\theta} \quad (9c)$$

This system can be linearized about an operating point, i.e., $b=\cos(\theta-\alpha)$. Even so, the control problem with the cost functional in Eq. (8) does not lend itself readily to a closed-form solution.

V. Arbitrary Attitude Angle at Impact

Since one weapon is used against a variety of targets, it would be desirable to be able to specify an arbitrary attitude angle δ_0 at impact. This paper has dealt with the case

$$\delta_0=0 \quad \text{(vertical impact)}$$

It will be shown that the resulting control law can be used to achieve any desired attitude angle at impact.

Using the same system dynamics, the problem is restated to design a controller u to minimize

$$J=Y_d^2(t_f)+\gamma[\theta(t_f)-\delta_0]^2+\beta\int_0^{t_f} u^2(t)\mathrm{d}t \quad (10)$$

subject to the constraints $|Y_d(t_f)|\leq 5$ ft, and $|\theta(t_f)-\delta_0|\leq 5°$.

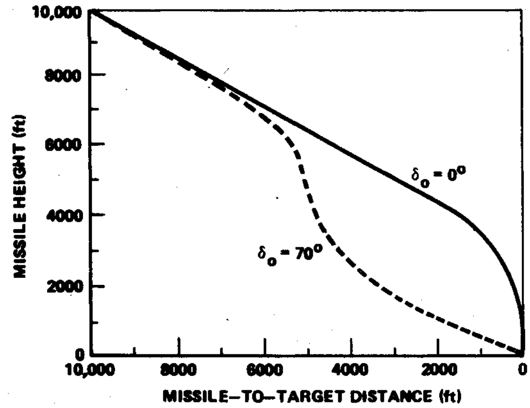

Fig. 4 Missile trajectories for $\delta_0=0°, 70°$.

Relating this problem to the one previously considered, a new state variable ϕ is introduced which is defined by

$$\phi(t)\equiv\theta(t)-\delta_0 \quad (11)$$

The dynamics (assuming no lag) become

$$\dot{Y}_d=\dot{Y}_d \quad (12a)$$

$$\ddot{Y}_d=(K_1/\omega_1)u\cos(\phi+\delta_0) \quad (12b)$$

$$\dot{\phi}=K_{\dot{\theta}}u \quad (12c)$$

and the cost functional to be minimized is similar to Eq. (10). This problem is not identical to the one solved, but, upon linearizing about the desired attitude conditions at impact, i.e., $\theta(t_f)=\delta_0$, the equation for $\ddot{Y}_d$ becomes

$$\ddot{Y}_d=-(K_1/\omega_1)bu \quad (b\equiv\cos\delta_0) \quad (13)$$

Symbolically, the problems are now identical (identify θ with ϕ), and it is asserted that the same control law will achieve an arbitrary attitude impact angle by suitably choosing γ β, and b. The assertion is strengthened by the results wherein the same control law with different parameter values achieved the two trajectories shown in Fig. 4, where $\delta_0=0°, 70°$.

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